AN INTRODUCTION TO THE FINITE ELEMENT METHOD FOR YOUNG ENGINEERS

PART TWO: 2D BEAM FORMULATIONS

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Primera

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I. Introduction

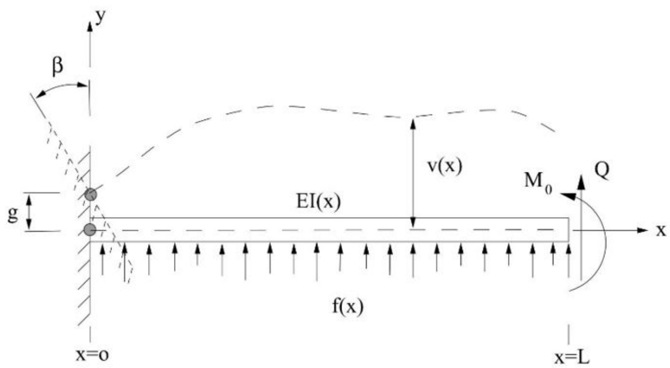
This article is the second in a series that will attempt to introduce some of the rich and complex theory that forms the foundation of the finite element method of analysis (FEM). The focus for this article is on beam formulations which in the author's opinion constitute the vast majority of FEM analysis conducted by practicing structural engineers. Although the current discussions will be limited to 2D Bernoulli Euler beam formulations, several topics that play a role in more advanced applications will be introduced. With a much clearer understanding of the limitations and key assumptions underlying beam formulations, it is the author's hope that practitioners will become more sophisticated users of commercial FEM software in regards to beam and frame analysis.

II. 2-D Example

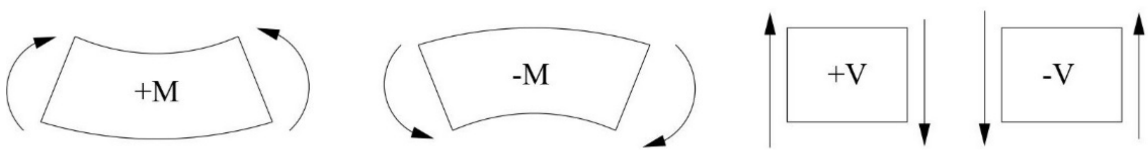
To help introduce basic mathematical concepts we will limit current discussion to a two-dimensional problem involving flexure of a beam under the action of a transverse distributed load. For current discussions shear deformations will be neglected, i.e. a Bernoulli-Euler formulation will be employed, and axial loading/deformation will not be included. Subsequent articles will introduce the effects of employing shear deformations, i.e. Timoshenko beam theory, and including axial forces/deformations before extending the formulations to 2D frames and eventually 3D frames. The article will conclude with a numerical example to tie together some of the key concepts introduced.

A. Model Problem and Sign Convention

For the model beam problem we will consider a beam spanning along the x direction from x=0 to x=L. The beam is acted upon by a transverse distributed load f(x) along its axis in addition to a concentrated moment M0 and transverse point load Q at the left end. The loading results in a transverse deflection in the y direction given by v(x) and a rotation of the beam cross section at any point x given by derivative of v(x), i.e. dv/dx, due to the Bernoulli-Euler formulation in which the cross sections will remain perpendicular to the neutral axis of the beam. The beam consists of an isotropic linear elastic material with a Young's Modulus of E and a geometric cross section with a moment of inertia I. At this point the flexural stiffness of the beam given by EI will be assumed to vary with location along the beam, i.e. EI=EI(x).



For all subsequent discussions we will adopt the following sign convention for the internal shear force V(x) and internal bending moment M(x) that develops in the beam:



Additionally, the transverse loading f(x) and transverse deflection v(x) are both positive if acting in the positive y direction. Concentrated loads are considered positive if acting in the positive y direction while concentrated moments and rotations are positive if counterclockwise.

For a well-defined 2-D Bernoulli-Euler beam problem with a unique solution, a total of four boundary conditions are required; two at each end. For the model problem the two boundary conditions at x=0 are called *displacement type* boundary conditions and are given by:

$$v(0) = g$$

$$\frac{dv}{dx} = \beta$$

in which g and β are prescribed constants.

At x=L, the two boundary conditions for the model problem are called *traction* or *force* type boundary conditions and are given by:

$$V(L) = -Q$$

$$M(L) = M_0$$

Commonly encountered boundary conditions for Bernoulli-Euler beams include:

- Fixed ends: v=0 and dv/dx=0, i.e. transverse displacements and rotations are precluded
- Simply supported ends: v=0 and M=0, transverse displacement is prevented and the internal bending moment is zero
- Free ends: V=0, M=0, both the internal shear force and bending moments are zero

Enforcing equilibrium and the Bernoulli-Euler assumption that cross sections will remain perpendicular to the neutral axis of the beam results in the following equations[1]

$$\frac{dV(x)}{dx} = f(x) \qquad \forall \ x \in 0, L$$

$$\frac{dM(x)}{dx} = V(x) \quad \forall \ x \ \in 0, L$$

$$M(x) = EI(x)\frac{d^2v(x)}{dx^2} \quad \forall \ x \ \in 0, L$$

Combining the above equations results in the differential equation of equilibrium that must be satisfied at every single point along the beam

$$\frac{d}{dx^2}\left[EI(x)\frac{d^2v(x)}{dx^2}\right] = f(x) \quad \forall \ x \ \in 0, L$$

As discussed in Article 1 of the series, if a finite difference approach where to be undertaken to approximate the solution to the model problem; approximations for the fourth order derivatives present in the above equation would be required.

B. Weak (Variational) Form of Model Problem

Prior to introducing finite element approximations for solving the model problem, the differential equation of equilibrium along with the prescribed boundary conditions need to be recast into a variational form. Derivation of the variational form (or weak form) will not be presented but the interested reader can consult the references listed at the end of the article. For the model problem, the variational form of the equation of equilibrium takes the form

[1] The mathematical symbol ∀ is shorthand notation denoting "for all and any" and ∈ is shorthand notation for "element in".

$$\int_0^L EI(x)\frac{d^2\omega(x)}{dx^2}\frac{d^2v(x)}{dx^2}dx = M_0\frac{d\omega(x)}{dx}\bigg|_{x=L} + Q\omega(L)$$
$$+\int_0^L f(x)\omega(x)dx \quad \forall\ \omega(x) \in W$$

To complete the variational form of the model problem, a definition for the space of admissible transverse displacements v(x) and admissible weighting functions (x) must be defined, T and W respectively.

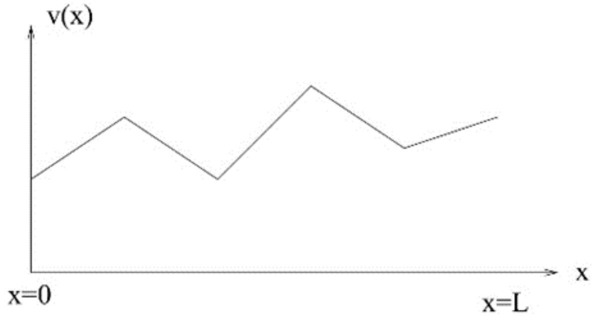
$$T = \left\{v(x)|v:(0,L) \to \mathbb{R}, v(x) \in C_b^1(0,L), v(0) = g, \frac{dv(x)}{dx}\bigg|_{x=0} = \beta\right\}$$

$$W = \left\{w(x)|w:(0,L) \to \mathbb{R}, w(x) \in C_b^1(0,L), w(x) = 0, \frac{dw(x)}{dx}\bigg|_{x=0} = 0\right\}$$

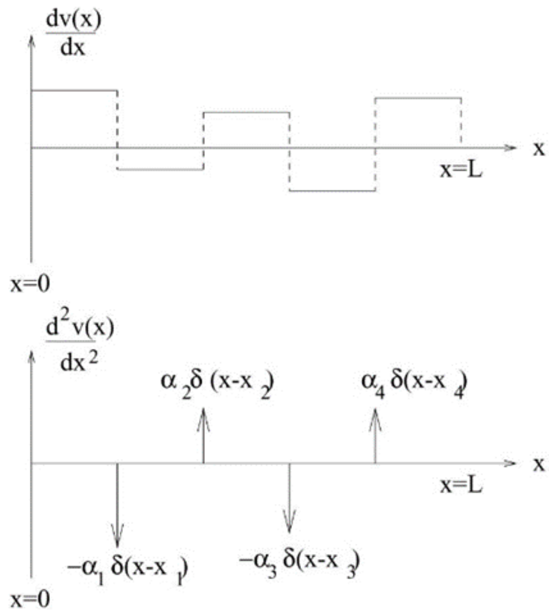
Before proceeding any further an explanation of the definitions for both the space of admissible displacements, T, and weighting functions, W, would be beneficial. The term v:(0,L) → ℝ simply states that the transverse displacement function v(x) accepts values of x between 0 and L and returns a real number. That real number physically signifies the transverse displacement at location x along the length of the beam. The term $v(x) \in C_b^1$ signifies that the transverse displacements are to be continuous and bounded. Furthermore, the first derivatives of the transverse displacements, i.e. $\frac{dv}{dx}$, are also required to be continuous and bounded. In general, a function f is said to be of class C_b^kcontinuity if its first k derivatives (k=0,1,...) are continuous and bounded, i.e. $\left|\frac{d^kf}{dx^k}\right| < \infty$. Because of the Bernoulli-Euler approximation, requiring the first derivative to be continuous means that the beam will not experience sudden jumps in the rotation of the cross sections along the axis of the beam. Finally, the last two terms in the definition of T require that the admissible transverse displacement functions must meet the displacement type boundary conditions a-priori. Similar descriptions apply to the definition of the space of admissible weighting functions, W, with the exception that the weighting functions are required to be equal to zero at any point where the transverse displacement is prescribed and the derivative of the weighting functions must be zero at any points where the slope of the beam is prescribed.

A few more observations on the variational form for the model problem:

- Consider the result of using linear piecewise continuous functions for both the transverse displacements and the weighting functions as was done with the 1-D model bar problem of article 1.



For the above piecewise linear approximations, the first and second derivatives of v(x) would have the following form:



Here $\delta(x - x_a)$ is the Dirac delta function given by

$$\delta(x - x_a) = \lim_{\varepsilon \to 0} \begin{cases} \dfrac{1}{2\varepsilon} & x_a - \varepsilon \le x \le x_a + \varepsilon \\ 0 & x < x_a - \varepsilon;\;\; x > x_a + \varepsilon \end{cases}$$

Additionally, the Dirac delta function has the following property

$$\int_{-\infty}^{+\infty} [\delta(x - x_a)]^2 dx = \infty$$

Therefore if both $\frac{dv}{dx}$ and $\frac{d\omega}{dx}$ experience discontinuities (jumps) at the same location x then the integral

$$\int_0^L EI(x)\frac{d^2\omega(x)}{dx^2}\frac{d^2v(x)}{dx^2}dx$$

becomes unbounded and thus undefined. The above once again illustrates why functions that are continuous and have continuous first derivatives will need to be employed for our approximations.

- The weak form is just a restatement of the principle of virtual work where the internal virtual work (neglecting shear and axial strain contributions) due to a virtual displacement ω(x) is given by

$$\int_0^L EI(x)\frac{d^2\omega(x)}{dx^2}\frac{d^2v(x)}{dx^2}dx$$

noting that $EI(x)\frac{d^2v(x)}{dx^2}$ corresponds to the internal bending moment at location x and that $\frac{d^2\omega(x)}{dx^2}$ is the virtual curvature of the beam at location x. The external virtual work is given by

$$M_0\frac{d\omega(x)}{dx}\bigg|_{x=L} + Q\omega(l) + \int_0^L f(x)\omega(x)dx$$

where the first term is the virtual work done by applied moment M0, the second term is the virtual work done by the applied transverse load Q, and the last term accounts for the virtual work performed by the applied distributed transverse load f(x).

C. FEM Formulation

Using the above variational form of the model problem under consideration, a finite element formulation can now be presented. The FEM formulation results from discretizing (breaking up) the beam into *n finite elements* as shown below:

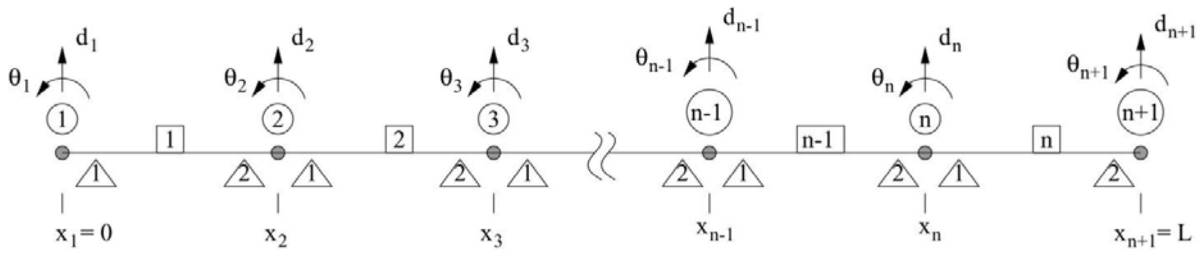


Figure 1: FEM discretization of model problem

The total length of the beam is broken into *n*, *n*=1, 2, 3, ... non overlapping elements labeled with an integer number enclosed in a box. At each element end there exists a node denoted by a number enclosed by a circle. For future discussions we will take an element by element view of the model problem. From the element point of view the right node is denoted by local node 1 and is enclosed in a triangle while the left node is denoted by local node 2 and is also enclosed in a triangle. Thus, global node 2 is the same as local node 2 for element 1 and local node 1 for element 2. At each node *i* we will have an unknown transverse displacement denoted by d_i along with a corresponding unknown rotation $\theta i=\frac{dv}{dx}\Big|_{x=x_i}$. Similarly, the weighting function (virtual transverse displacement) will have a value at node *i* which is denoted by ωi along with a corresponding unknown derivative (i.e. virtual beam rotation) at node *i* which is denoted by $\varphi i=\frac{dv(x_i)}{dx}\Big|_{x=x_i}$. Therefore at each node *two degrees of freedom* (dof) exist, a transverse displacement and a rotation. Since each element has two nodes, one at each end, each element has a total of 4 dofs (2 per node).

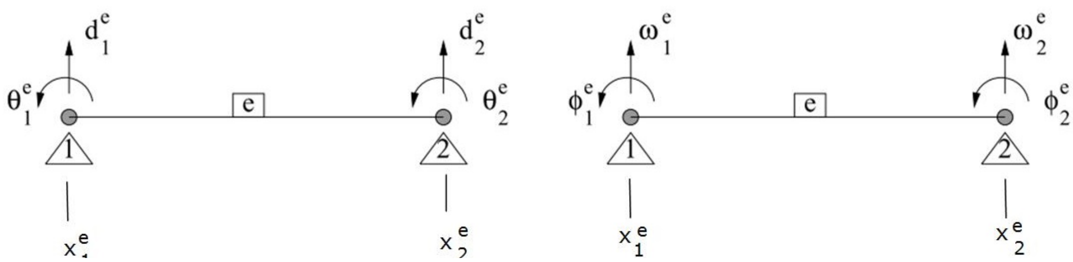


Figure 2: Element Description

1. Notation

Before proceeding any further a definition of the notation that will be employed from this point on will be useful. At times we will take a global view of the model problem, that is the entire set of elements and nodes will be taken as a whole. In this point of view we will denote global nodes by the labels enclosed in the circles and denote particular elements using the labels enclosed in the boxes as shown on Figure 1. Global displacements at a particular node *i* will be denoted by di and the corresponding rotation by θ_i. Similarly, the virtual global displacements at the same node *i* will be denoted by ω_i and the corresponding virtual rotation by φ_i.

Another important point of view to be used in the following discussion is within an element *e*. Figure 2 shows the element point of view for the nodal displacements/rotations and the nodal virtual displacements/rotations. When discussing quantities at the element level, a superscript *e* will be employed to denote any element quantity associated with a general element *e*, while the element integer label will be used as a superscript to denote quantities associated with that particular element.

Finally, as will be seen shortly approximations to the transverse displacement function v(x) and the virtual displacement function ω(x) will be constructed. The approximations will be tied directly to the underlying *finite element mesh* (system of nodes and elements) that discretizes the problem. That is; change the underlying fem mesh by increasing/decreasing the number of elements employed and/or moving the node locations around and the approximations employed will be altered. We will make use of the notation that he represents the length of an element *e*. Therefore to underscore the relationship between our approximations and the fem mesh we employ, our approximations will be denoted as follows:

$$v(x) \approx v^h(x)$$

$$\omega(x) \approx \omega^h(x)$$

2. Element Shape Functions

We have already shown that simple piecewise linear approximations will not work for the model beam problem. Instead approximations that are continuous and have continuous first order derivatives are required. To derive the *Cubic Hermitian Shape Functions* commonly employed in FEM beam formulations, let us consider a counterpart to the differential equation of equilibrium for the model problem in which EI(x) is replaced with a constant average value within each element and the transverse distributed load f(x) is taken as zero. With these simplifications the differential equation of equilibrium takes the following form within each element

$$\overline{EI}\frac{d^4v^h(x)}{dx^4} = 0 \quad \forall\, x \in x_1^e \leq x \leq x_2^e$$

Here x_1^e is the x coordinate of local node 1 for the element and x_2^e is the x coordinate of local node 2 for the element. A general solution for the above equation within each element takes the form

$$v^h(x) = C_0x^3 + C_1x^2 + C_2x + C_3$$

where $v^h(x)$ is an approximation to the actual transverse displacement v(x) and C_0, C_1, C_2, and C_3 are constants of integration that can be solved for by enforcing values of v^h and $\frac{dv^h}{dx}$ at specific points along the element. To force our approximation to match the displacements and rotations at each end of the element we require the following equations to hold:

$$C_0(x_1^e)^3 + C_1(x_1^e)^2 + C_2x_1^e + C_3 = d_1^e$$

$$C_0(x_2^e)^3 + C_1(x_2^e)^2 + C_2x_2^e + C_3 = d_2^e$$

$$3C_0(x_1^e)^2 + 2C_1x_1^e + C_2 = \theta_1^e$$

$$3C_0(x_2^e)^2 + 2C_1x_2^e + C_2 = \theta_2^e$$

Solving the above system of linear equations for the constants of integration and rearranging terms leads to the following approximation for the transverse displacement within each element

$$v^h(x) = N_1^e(x)d_1^e + N_2^e(x)\theta_1^e + N_3^e(x)d_2^e + N_4^e(x)\theta_2^e$$

where the element shape functions are given by

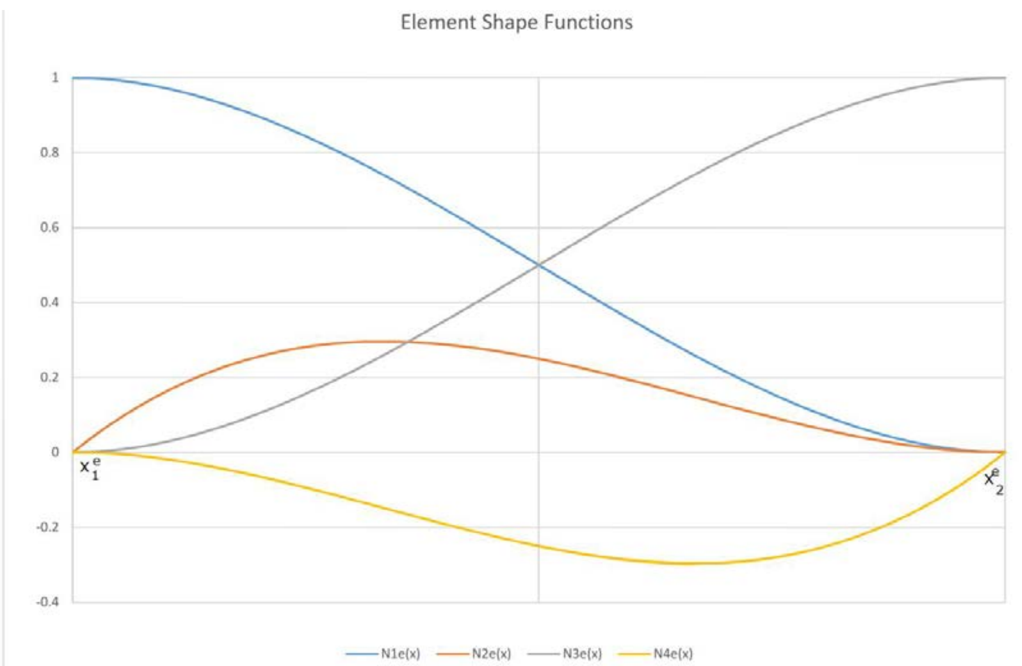
$$N_1^e = \frac{1}{(h^e)^3}(x - x_2^e)\left(2x^2 - x_2^e x - 3x_1^e x - x_2^e(x_2^e - 3x_1^e)\right)$$

$$N_2^e = \frac{1}{(h^e)^2}(x - x_1^e)(x - x_2^e)^2$$

$$N_3^e = \frac{1}{(h^e)^3}(x - x_1^e)\left(-2x^2 + x_1^e x + 3x_2^e x + x_1^e(x_1^e - 3x_2^e)\right)$$

$$N_4^e = \frac{1}{(h^e)^2}(x - x_2^e)(x - x_1^e)^2$$

$h^e = x_2^e - x_1^e$ is the element length



Upon more careful observation, the reader will note that all shape functions are zero at x_1^e except for N_1^e which is exactly equal to one at that location. Similarly, all shape functions are equal to zero at x_2^e except for N_3^e which is exactly equal to one at that location. Likewise, all shape functions have slopes that are equal to zero at x_1^e except for N_2^e which has a slope that is exactly equal to one at that location. All shape functions have slopes that are equal to zero at x_2^e except for N_4^e which has a slope that is exactly equal to one at that location. In this way our approximation will result in displacements and rotations at the ends of the element that will match the nodal displacements and nodal rotations at each end.

3. Nodal Shape Functions

We have already seen that an internal node will be shared between two adjacent elements such that the node will be identical to local node 2 for the element on the right and to local node 1 for the element on the left. Combining the element shape functions for adjacent elements that share a node, we can obtain the shape functions associated with each node. These functions are called the *nodal shape functions*. If we denote the element on the right by the label *e* and the element on the left by *e+1* we obtain for the nodal shape functions at node A (the shared node), the following:

$$N_A(x) = \begin{cases} N_3^e(x) & x_{A-1} \leq x < x_A \\ N_1^{e+1}(x) & x_A \leq x \leq x_{A+1} \\ 0 & \text{otherwise} \end{cases}$$

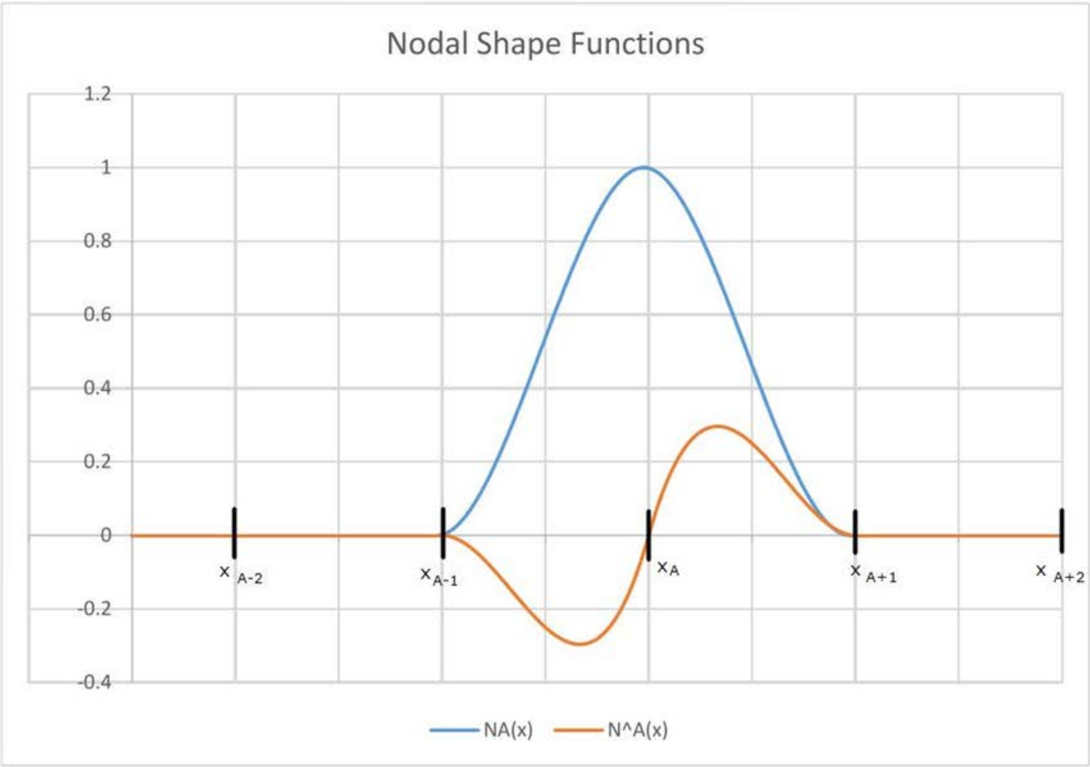
$$\widehat{N}_A(x) = \begin{cases} N_4^e(x) & x_{A-1} \leq x < x_A \\ N_2^{e+1}(x) & x_A \leq x \leq x_{A+1} \\ 0 & \text{otherwise} \end{cases}$$

Here there are two shape functions at each node A. Shape function NA is used to interpolate the transverse displacements at each node while shape function $\widehat{N}_A$ is used to interpolate the rotations at each node. Taking a global view of our approximations (for the entire beam) we then obtain the following approximation to our transverse displacement function and weighting function, respectively

$$v^h(x) = \sum_{A=1}^{n+1} N_A(x) d_A + \sum_{A=1}^{n+1} \widehat{N}_A(x) \theta_A$$

$$\omega^h(x) = \sum_{A=1}^{n+1} N_A(x) \omega_A + \sum_{A=1}^{n+1} \widehat{N}_A(x) \phi_A$$

Plotting the nodal shape functions we obtain



A few observations on the nodal shape functions:

- Nodal shape function $N_A(x)$ is exactly equal to 1 at node A but is zero at all other nodes. Furthermore, the nodal shape function $\widehat{N}_A(x)$ is equal to zero at all nodes including node A. Thus the transverse displacement at x=xA will be exactly equal to d_A, the nodal displacement at node A.
- The slopes of nodal shape function $N_A(x)$ is exactly equal to 0 at all nodes including node A. Furthermore, the nodal shape function $\widehat{N}_A(x)$ has slope equal to 1 at $x=x_A$ and zero slope at all other nodes. Thus the rotation at $x=x_A$ will be exactly equal to θ_A, the nodal rotation at node A.
- The shape functions are continuous and have continuous first derivatives as required to obtain solutions that are bounded.
- The shape functions are local approximations in that each shape function is non zero only over a specific region. The locality of the approximations used will lead to a

specific structure to the stiffness matrix that can be exploited within an algorithm to obtain the fem solution.

4. Element Stiffness Matrices and Force Vectors

Now with an approximation in hand for the transverse displacement and weighting functions, we can return our attention to the variational form of the model problem. Consider the internal virtual work term given by

$$\int_0^L EI(x)\frac{d^2\omega^h(x)}{dx^2}\frac{d^2v^h(x)}{dx^2}dx$$

Since we have constructed our approximations on an element by element basis, it would be convenient to split the integrand into a summation of integrands calculated within each element domain, i.e.

$$\sum_{e=1}^{n}\int_{x_e}^{x_{e+1}} EI(x)\frac{d^2\omega^h(x)}{dx^2}\frac{d^2v^h(x)}{dx^2}dx$$

Within each element we rewrite our approximations in matrix form as follows

$$v^h(x) = [N_1^e,\ \ N_2^e,\ \ N_3^e,\ \ N_4^e\ ]_{1x4}\begin{pmatrix} d_1^e \\ \theta_1^e \\ d_2^e \\ \theta_2^e \end{pmatrix}_{4x1} = \mathbb{N}_{1x4}^e\vec{d}_{4x1}^e$$

$$w^h(x) = [N_1^e,\ \ N_2^e,\ \ N_3^e,\ \ N_4^e\ ]_{1x4}\begin{pmatrix} w_1^e \\ \phi_1^e \\ w_2^e \\ \phi_2^e \end{pmatrix}_{4x1} = \mathbb{N}_{1x4}^e\vec{\omega}_{4x1}^e$$

Here the subscripts denote the size of the individual matrices, shown for clarity, with the notation that *mxn* indicates a matrix with *m* rows and *n* columns. $\vec{d}_{4x1}^e$ and $\vec{\omega}_{4x1}^e$ are the 4x1 element displacement vector and element virtual displacement vector, respectively. We also introduce an *element strain displacement* matrix $\mathbb{B}^e$ that relates the curvature in the beam to the element displacement vector such that

$$\mathbb{B}^e(x) = \left[\frac{d^2}{dx^2}N_1^e, \frac{d^2}{dx^2}N_2^e, \frac{d^2}{dx^2}N_3^e, \frac{d^2}{dx^2}N_4^e\ \right]_{1x4}$$

and

$$\frac{d^2v^h}{dx^2} = \mathbb{B}_{1x4}^e\vec{d}_{4x1}^e$$

$$\frac{d^2\omega^h}{dx^2} = \mathbb{B}^e_{1x4}\vec{\omega}^e_{4x1} = (\vec{\omega}^e)^T_{1x4}\mathbb{B}^{eT}_{4x1}$$

The internal virtual work term now becomes

$$\sum_{e=1}^{n}\int_{x_e}^{x_{e+1}} EI(x)(\vec{\omega}^e)^T_{1x4}\mathbb{B}^{eT}_{4x1}\mathbb{B}^e_{1x4}\vec{d}^e_{4x1}dx$$

Or, since the element displacement vector and element virtual displacement vector do not depend on location x

$$\sum_{e=1}^{n}(\vec{\omega}^e)^T_{1x4}\left\{\int_{x_e}^{x_{e+1}} EI(x)\mathbb{B}^{eT}_{4x1}\mathbb{B}^e_{1x4}dx\right\}\vec{d}^e_{4x1}$$

Introducing the *element stiffness matrix* $\mathbb{k}^e$ that is a 4x4 matrix as

$$\mathbb{k}^e_{4x4} = \int_{x_e}^{x_{e+1}} EI(x)\mathbb{B}^{eT}_{4x1}\mathbb{B}^e_{1x4}dx$$

The internal virtual work term now simplifies to

$$\sum_{e=1}^{n}(\vec{\omega}^e)^T_{1x4}\{\mathbb{k}^e_{4x4}\}\vec{d}^e_{4x1}$$

For the common case of EI(x) being equal to a constant value, i.e. a prismatic beam the element stiffness matrix takes the following form

$$\mathbb{k}^e_{4x4} = \frac{EI}{h^e}\begin{bmatrix} \frac{12}{(h^e)^2} & \frac{6}{h^e} & \frac{-12}{(h^e)^2} & \frac{6}{h^e} \\ \frac{6}{h^e} & 4 & \frac{-6}{h^e} & 2 \\ \frac{-12}{(h^e)^2} & \frac{-6}{h^e} & \frac{12}{(h^e)^2} & \frac{-6}{h^e} \\ \frac{6}{h^e} & 2 & \frac{-6}{h^e} & 4 \end{bmatrix}$$

The external virtual work due to the applied distributed transverse load f(x) is then given by

$$\sum_{e=1}^{n}\int_{x_e}^{x_{e+1}} f(x)\omega^h(x)dx = \sum_{e=1}^{n}(\vec{\omega}^e)^T_{1x4}\vec{f}^e_{4x1}$$

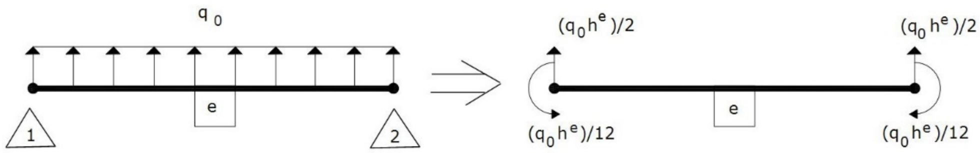
where the *element force vector* $\vec{f}^e_{4x1}$ is given by

$$\vec{f}^e_{4x1} = \int_{x_e}^{x_{e+1}} f(x)(\mathbb{N}^e)^T_{4x1}dx$$

Before proceeding further let's consider the format for the element force vector for specific cases of transverse loading.

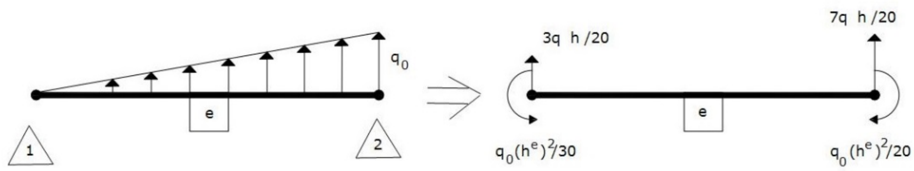
- Uniform Distributed load f(x)=q_0

$$\vec{f}^{e}_{4x1} = \frac{q_0 h^e}{2}\begin{bmatrix} 1 \\ \frac{h^e}{6} \\ 1 \\ -\frac{h^e}{6} \end{bmatrix}$$



- Triangular distributed load $f(x) = \frac{q_0}{h^e}(x - x_1^e)$

$$\vec{f}^{e}_{4x1} = \frac{q_0 h^e}{2}\begin{bmatrix} \frac{3}{10} \\ \frac{h^e}{15} \\ \frac{7}{10} \\ -\frac{h^e}{10} \end{bmatrix}$$



On closer inspection of the results for the above two cases, one can see that the applied distributed load is in essence replaced with a statically equivalent set of nodal forces acting at the ends of the element.

The external virtual work also consists of two additional terms involving the applied concentrated moment M_0 and concentrated transverse load Q acting at the end of the beam located at x=L. It will be best to treat these terms on a global basis once the element equations are assembled as discussed shortly for the full beam. Before leaving the element point of view for our model problem we need to consider how to handle the effects of non-zero prescribed displacements and/or rotations at specific locations (i.e. nodes) along the beam. For the model problem both the transverse displacement and rotation have specified prescribed values at x=0 equal to g and β; respectively. Most elements will not have any prescribed displacements or rotations at either end with only a few elements being affected by prescribed non-zero displacements/rotations. For the model problem currently under consideration, only element 1 is affected by non-zero prescribed displacements/rotations.

Consider an element *e* in which some of the displacements and/or rotations at either end are prescribed to have a non-zero value. This situation arises when a support point along the beam experiences a known settlement or rotation. The displacement vector for the element can be split into a sum of two vectors, a vector containing only the unknown degrees of freedom for the element, $\hat{d}^e_{4x1}$, and a vector containing the non-zero prescribed values, $d^e_{g\,4x1}$

$$\vec{d}^e_{4x1} = \hat{d}^e_{4x1} + d^e_{g\,4x1}$$

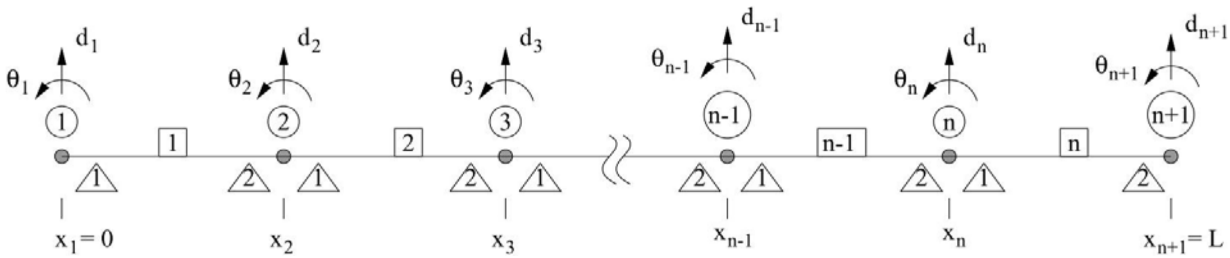
With this notation in place we can rewrite the internal virtual work term as

$$\sum_{e=1}^{n} (\vec{\omega}^e)^T_{1x4}\left[\{\mathbb{k}^e_{4x4}\}\hat{d}^e_{4x1} + \{\mathbb{k}^e_{4x4}\}d^e_{g\,4x1}\right] = \sum_{e=1}^{n} (\vec{\omega}^e)^T_{1x4}\{\mathbb{k}^e_{4x4}\}\hat{d}^e_{4x1} + \sum_{e=1}^{n} (\vec{\omega}^e)^T_{1x4}\left(-\vec{f}^{\,e}_{g\,4x1}\right)$$

Here $\vec{f}^e_g = -\{\mathbb{k}^e_{4x4}\}d^e_{g\,4x1}$ is an element force vector emanating from the non-zero prescribed displacements. The negative sign has been added for convenience as this term will be lumped together with the external virtual work terms upon assembling the global system of equations for the model problem.

5. Assembly of the Global System of Equations

To obtain an approximation to our model problem using the fem formulation presented, a global system of equations must be assembled that account for all the degrees of freedom. Recall that we have made use of a system of finite elements and global nodes for the entire beam as follows:



The *direct stiffness method* is an efficient method for assembling the global system of equations making use of the element by element contributions to both the internal and external virtual work terms. Furthermore, it is a method that can easily be implemented as a computer algorithm. The interested reader is referred to the first two references for more details. In this article only the main ideas will be presented.

As a first step to the direct stiffness method, the displacements and rotations are renumbered such that the lowest numbered global node will have its displacement and rotation numbered sequentially followed by the next lowest numbered global node and so on. If a displacement or a rotation at a node is prescribed then that dof is not numbered but is instead assigned a label of 0. Once all the displacements and rotations have been numbered by the preceding procedure the largest integer number will represent the total number of degrees of freedom for the entire problem, i.e. the total number of unknown displacements/rotations for the problem, denoted by *neq*. Furthermore, at each element e a mapping array, l_m^e, will be used to map (link) the elements dofs to the globally numbered degrees of freedom.

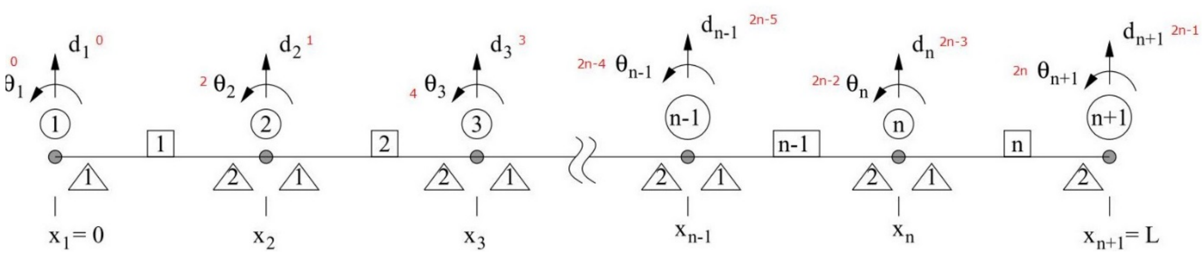


Figure 3: Global dofs labelling

For the current model problem the displacement and rotation at global node 1 are not numbered since both have been prescribed values. The displacement at global node 2 is numbered 1 while the rotation at global node 2 is numbered 2 and so on. When all the nodal displacements and rotations have been numbered there will be a total of *neq*=2*n equations (dofs) where n represents the integer number of elements used. The mapping arrays for the 1st, 2nd, *n*th-1, and *n*th elements are given by

$$l_m^1 = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{Bmatrix} \quad l_m^2 = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{Bmatrix} \quad l_m^{n-1} = \begin{Bmatrix} 2n-5 \\ 2n-4 \\ 2n-3 \\ 2n-2 \end{Bmatrix} \quad l_m^n = \begin{Bmatrix} 2n-3 \\ 2n-2 \\ 2n-1 \\ 2n \end{Bmatrix}$$

Once the element contributions to the global system of equations is completed the variational form of the model problem will take the following shape

$$\omega^T{}_{1xneq}\left(\mathbb{K}_{neqxneq}\vec{d}_{neqx1} - \vec{F}_{neqx1}\right) = 0 \qquad \forall\, \omega_{neqx1}$$

Where ω_{neqx1} is the *global virtual displacement* vector that contains one row for each global dof, $\mathbb{K}_{neqxneq}$ is the *global stiffness matrix*, and $\vec{F}_{neqx1}$ is the global force vector that has *neq* rows. Since the above equation must hold for any global virtual displacement vector ω, the following equation must also hold

$$\mathbb{K}_{neqxneq}\vec{d}_{neqx1} = \vec{F}_{neqx1}$$

The global displacement vector for the model problem takes the following form where it should be noted that only displacements and rotations that are not prescribed are included:

$$\vec{d}_{neqx1} = \begin{Bmatrix} d_2 \\ \theta_2 \\ d_3 \\ \theta_3 \\ \vdots \\ \theta_n \\ d_{n+1} \\ \theta_{n+1} \end{Bmatrix}$$

The global stiffness matrix $\mathbb{K}$ is assembled by creating a zero matrix of size *neq* x *neq* with all entries equal to zero. The individual element stiffness matrices are then added to the global stiffness matrix employing the following algorithm:

```
For each element e from 1 to n:
    For each row i from 1 to 4:
        I=l^e_m(i)    (global index for local row i)
        If (I ≠ 0) then
            For each column j from 1 to 4:
                J=l^e_m(j)     (global index for local column j)
                If (J ≠ 0) then
                    𝕂(I,J)= 𝕂 (I,J)+𝕜^e(i,j)
                End if
            End loop over j
        End if
    End loop over i
End loop over elements
```

The global force vector $\vec{F}_{neqx1}$ is assembled by creating a zero vector of size *neq* x 1 with all entries equal to zero. The individual element force vectors due to the applied transverse load, f^e_{4x1}, and non-zero prescribed displacements, ${f_g}^e_{4x1}$, are then added to the global force vector employing the following algorithm:

```
For each element e from 1 to n:
        For each row i from 1 to 4:
                I=lm^e(i)      (global index for row i)
                If (I≠0) then
                        F(I) = F(I) + f^e(i) + f_g^e(i)
                End if
        End loop over i
End loop over elements
```

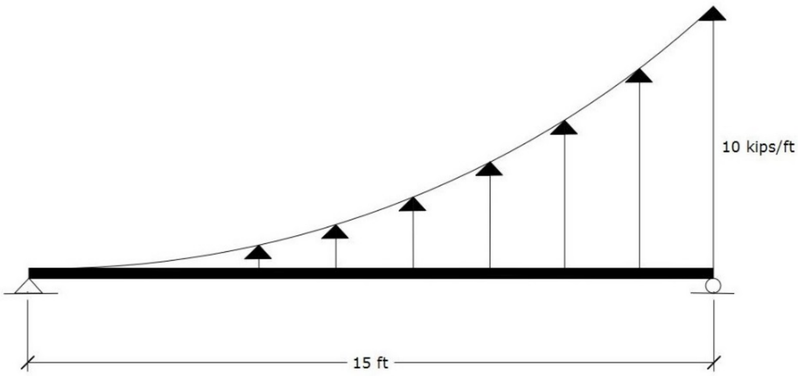
As a final step to assembling the global force vector, any applied concentrated loads and moments are added to the appropriate rows in the global force vector. For the model problem under consideration the value of Q, the applied transverse concentrated load at x=L, is added to the 2*n*-1 row of the global force vector; while the value of M_0 is added to the 2n^{th} row of the global force vector.

$$\vec{F}_{neqx1} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{2n-1} + Q \\ F_{2n} + M_0 \end{Bmatrix}$$

Once the global matrices are assembled, the system of equations can be solved using a numerical method such as Gaussian Elimination.

D. Assembly of the Global System of Equations

To highlight the previous topics covered we will consider an example problem of a simply supported prismatic beam acted upon by an exponential transverse load as shown below.



$$EI = constant = 4 \times 10^6\ kip \cdot ft^2$$

$$f(x) = \frac{q_0}{(e^1 - 1)}\left(e^{\left(\frac{x}{L}\right)} - 1\right)$$

$$q_0 = 10\frac{kip}{ft} \qquad L = 15\ ft$$

The differential equation of equilibrium has a known solution for the transverse displacement given by

$$v(x) = \alpha\left[e^{\left(\frac{x}{L}\right)} - \frac{1}{24}\left(\frac{x}{L}\right)^4 + \left(\frac{3}{12} - \frac{e^1}{6}\right)\left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L}\right)^2 + \left(\frac{31}{24} - \frac{5e^1}{6}\right)\left(\frac{x}{L}\right) - 1\right]$$

where

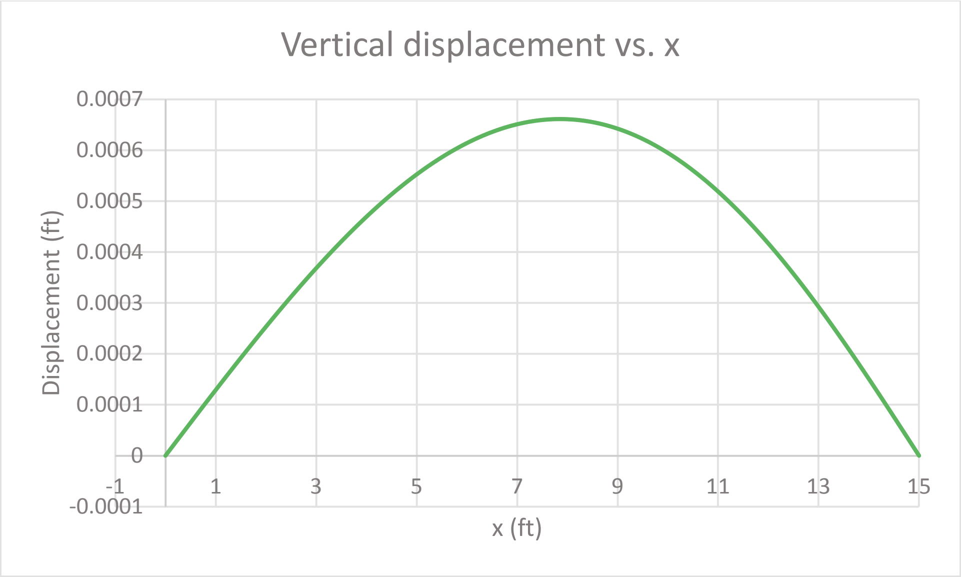
$$\alpha = \frac{q_0 L^4}{EI(e^1 - 1)} = 0.07366 \text{ ft}$$

The internal bending moment in the beam is given by

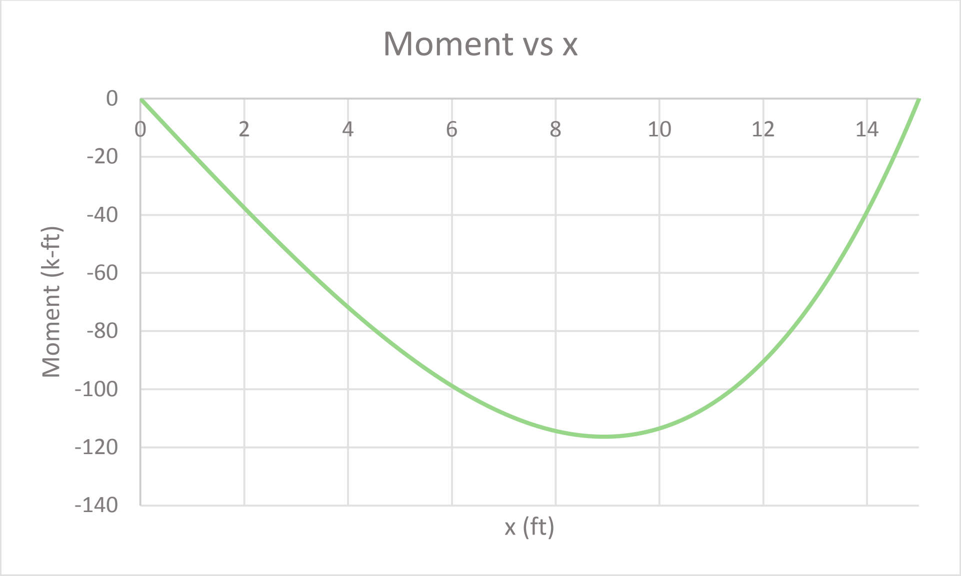
$$M(x) = EI\frac{d^2 v(x)}{dx^2} = \frac{q_0 L^2}{(e^1 - 1)}\left[e^{\left(\frac{x}{L}\right)} - \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{3}{2}\left(\frac{x}{L}\right) - e^1\left(\frac{x}{L}\right) - 1\right]$$

while the internal shear force in the beam is given by

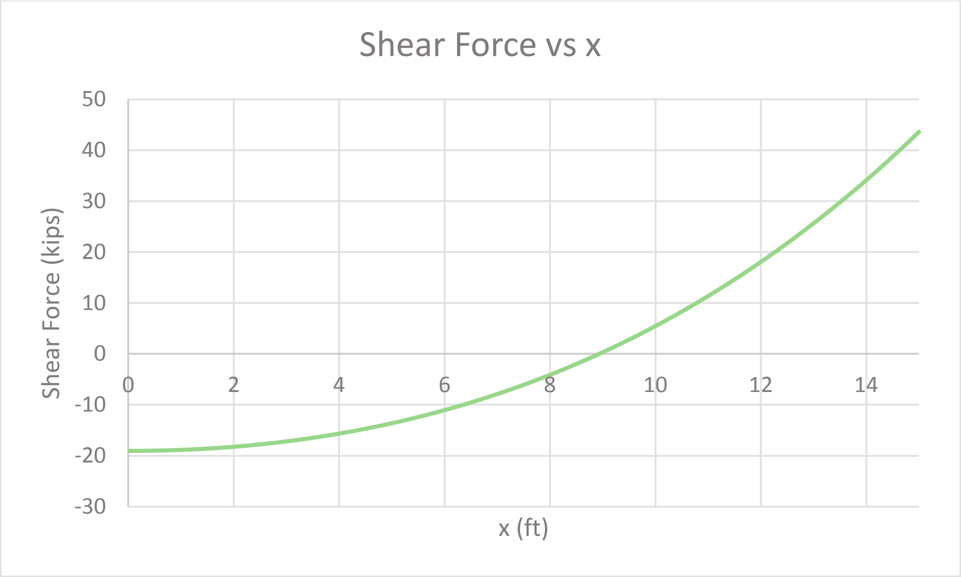
$$V(x) = \frac{dM(x)}{dx} = \frac{q_0 L}{(e^1 - 1)}\left[e^{\left(\frac{x}{L}\right)} - \left(\frac{x}{L}\right) + \frac{3}{2} - e^1\right]$$



Vertical displacement vs. x
Displacement (ft)
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0
-0.0001
-1
1
3
5
7
9
11
13
15
x (ft)



Moment vs x
Moment (k-ft)
0
-20
-40
-60
-80
-100
-120
-140
0
2
4
6
8
10
12
14
x (ft)



The example problem will be discretized with 3 elements of equal length. Due to the prescribed displacements at x=0 and x=L; the total number of global dofs are equal to *neq*=6 and are labeled in red in Figure 4.

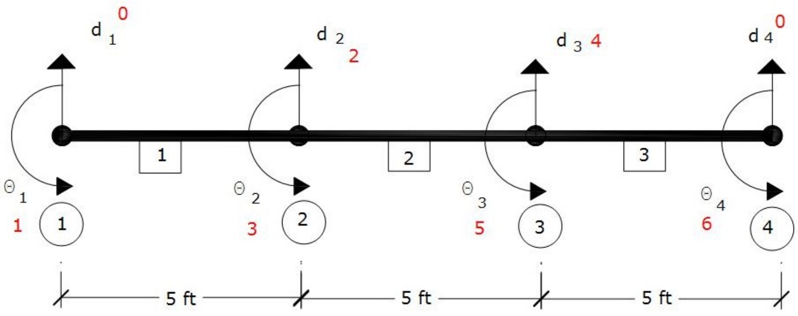


Figure 4: FE mesh with global dofs labeled in red.

The three element stiffness matrices are identical and equal to:

$$\mathbb{k}^1 = \mathbb{k}^2 = \mathbb{k}^3 = \begin{bmatrix} 3.84 \times 10^5 kip/ft & 9.6 \times 10^5 kip & -3.84 \times 10^5 kip/ft & 9.6 \times 10^5 kip \\ 9.6 \times 10^5 kip & 3.2 \times 10^6 kip \cdot ft & -9.6 \times 10^5 kip & 1.6 \times 10^6 kip \cdot ft \\ -3.84 \times 10^5 kip/ft & -9.6 \times 10^5 kip & 3.84 \times 10^5 kip/ft & -9.6 \times 10^5 kip \\ 9.6 \times 10^5 kip & 1.6 \times 10^6 kip \cdot ft & -9.6 \times 10^5 kip & 3.2 \times 10^6 kip \cdot ft \end{bmatrix}$$

The element force vectors due to the applied transverse load are given by:

$$f^1 = \begin{Bmatrix} 1.570\ kip \\ 1.760\ kip \cdot ft \\ 3.867\ kip \\ -2.717\ kip \cdot ft \end{Bmatrix} \qquad f^2 = \begin{Bmatrix} 7.946\ kip \\ 7.253\ kip \cdot ft \\ 11.15\ kip \\ -8.589\ kip \cdot ft \end{Bmatrix} \qquad f^3 = \begin{Bmatrix} 16.846\ kip \\ 14.92\ kip \cdot ft \\ 21.32\ kip \\ -16.783\ kip \cdot ft \end{Bmatrix}$$

Employing the direct stiffness matrix to assemble the global system of equations we obtain after assembling element 1 the following:

$$\mathbb{K} = \begin{bmatrix} 3.2\times 10^6 kip\cdot ft & -9.6\times 10^5 kip & 1.6\times 10^6\ kip\cdot ft & 0 & 0 & 0 \\ -9.6\times 10^5 kip & 3.84\times 10^5 kip/ft & -9.6\times 10^5 kip & 0 & 0 & 0 \\ 1.6\times 10^6\ kip\cdot ft & -9.6\times 10^5 kip & 3.2\times 10^6 kip\cdot ft & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad F = \begin{Bmatrix} 1.76\ kip\cdot ft \\ 3.867\ kip \\ -2.717\ kip\cdot ft \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

After assembling element 2 the global matrices become:

$$\mathbb{K} = \begin{bmatrix} 3.2\times 10^6 kip\cdot ft & -9.6\times 10^5 kip & 1.6\times 10^6\ kip\cdot ft & 0 & 0 & 0 \\ -9.6\times 10^5 kip & 7.68\times 10^5 kip/ft & 0 & -3.84\times 10^5 kip/ft & 9.6\times 10^5 kip & 0 \\ 1.6\times 10^6\ kip\cdot ft & 0 & 6.4\times 10^6 kip\cdot ft & -9.6\times 10^5 kip & 1.6\times 10^6\ kip\cdot ft & 0 \\ 0 & -3.84\times 10^5 kip/ft & -9.6\times 10^5 kip & 3.84\times 10^5 kip/ft & -9.6\times 10^5 kip & 0 \\ 0 & 9.6\times 10^5 kip & 1.6\times 10^6\ kip\cdot ft & -9.6\times 10^5 kip & 3.2\times 10^6 kip\cdot ft & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{Bmatrix} 1.76\ kip\cdot ft \\ 11.814\ kip \\ 4.536\ kip\cdot ft \\ 11.153\ kip \\ -8.589\ kip\cdot ft \\ 0 \end{Bmatrix}$$

After assembling element 3 we obtain final version of the global matrices due to the fact that there are no non-zero prescribed displacements or concentrated forces/moments applied:

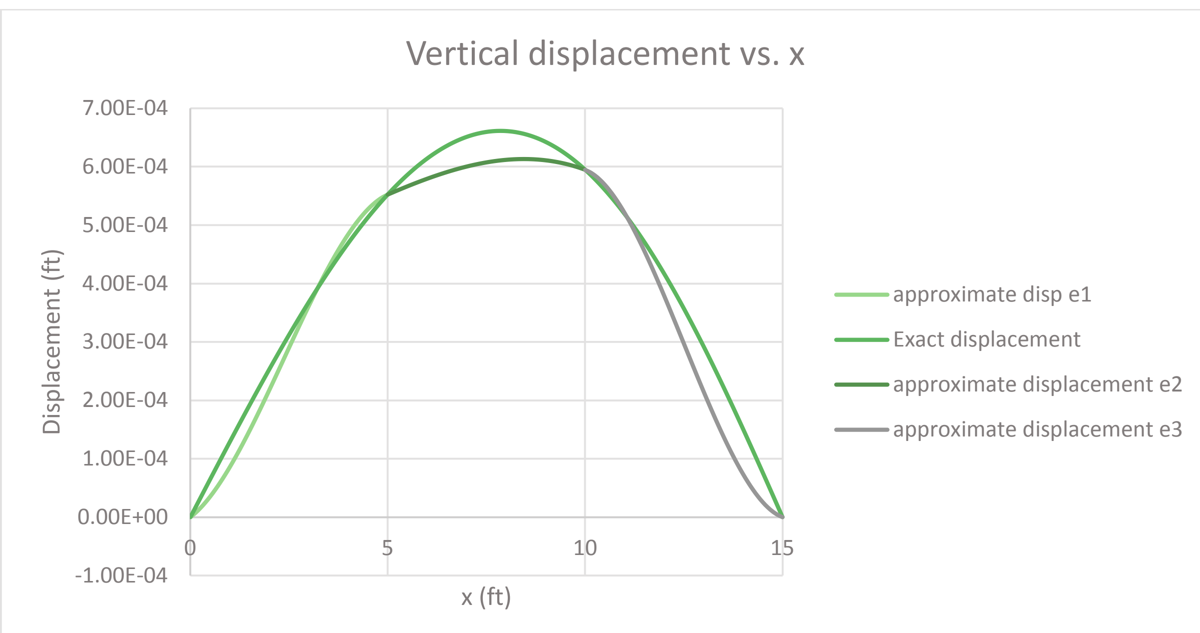
$$\mathbb{K} = \begin{bmatrix} 3.2\times10^6 kip\cdot ft & -9.6\times10^5 kip & 1.6\times10^6\ kip\cdot ft & 0 & 0 & 0 \\ -9.6\times10^5 kip & 7.68\times10^5 kip/ft & 0 & -3.84\times10^5 kip/ft & 9.6\times10^5 kip & 0 \\ 1.6\times10^6\ kip\cdot ft & 0 & 6.4\times10^6 kip\cdot ft & -9.6\times10^5 kip & 1.6\times10^6\ kip\cdot ft & 0 \\ 0 & -3.84\times10^5 kip/ft & -9.6\times10^5 kip & 7.68\times10^5 kip/ft & 0 & 9.6\times10^5\ kip \\ 0 & 9.6\times10^5 kip & 1.6\times10^6\ kip\cdot ft & 0 & 6.4\times10^6 kip\cdot ft & 1.6\times10^6\ kip\cdot ft \\ 0 & 0 & 0 & 9.6\times10^5\ kip & 1.6\times10^6\ kip\cdot ft & 3.2\times10^6 kip\cdot ft \end{bmatrix}$$

$$F = \begin{Bmatrix} 1.76\ kip\cdot ft \\ 11.814\ kip \\ 4.536\ kip\cdot ft \\ 28\ kip \\ 6.331\ kip\cdot ft \\ -16.783\ kip\cdot ft \end{Bmatrix}$$

Solving for the global displacements we obtain:

$$\vec{d} = \mathbb{K}^{-1}\vec{F} = \begin{Bmatrix} \theta_1 \\ d_2 \\ \theta_2 \\ d_3 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 1.298\times10^{-4} rad \\ 5.524\times10^{-4}\ ft \\ 7.295\times10^{-5}\ rad \\ 5.947\times10^{-4}\ ft \\ -6.193\times10^{-5}\ rad \\ -1.527\times10^{-4} rad \end{Bmatrix}$$

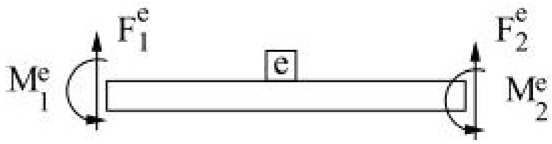
Due to the boundary conditions we also have $d_1 = d_4 = 0$. The resulting approximation to the transverse displacement vs the exact solution is shown below.



1. Assembly of the Global System of Equations

Having obtained a solution for the global displacements the element displacements vectors are now used to determine the resultant forces and moments at each end of the element via the equation

$$\vec{F}^e = \mathbb{k}^e_{4x4}\vec{d}^e_{4x1} - \vec{f}^e_{4x1} = \begin{Bmatrix} F^e_1 \\ M^e_1 \\ F^e_2 \\ M^e_2 \end{Bmatrix}$$



Since we have the relationships $\frac{dM(x)}{dx} = V(x)$ and $\frac{dV(x)}{dx} = f(x)$ we can make use of the element shape functions to obtain approximations to the shear force and bending moment within each element by making use of the nodal values. The approximation to the internal shear force within the element is given by:

$$V^h(x) = [N^e_1,\ N^e_2,\ N^e_3,\ N^e_4\ ]_{1x4} \begin{pmatrix} F^e_1 \\ f(x^e_1) \\ -F^e_2 \\ f(x^e_2) \end{pmatrix}_{4x1}$$

The approximation to the internal bending moment within the element is given by:

$$M^h(x) = [N^e_1,\ N^e_2,\ N^e_3,\ N^e_4\ ]_{1x4} \begin{pmatrix} -M^e_1 \\ F^e_1 \\ M^e_2 \\ -F^e_2 \end{pmatrix}_{4x1}$$

For the current numerical example the element resultant forces are given by:

a) Element 1

$$\vec{d}^1_{4x1} = \begin{Bmatrix} 0 \\ 1.298 \times 10^{-4} rad \\ 5.524 \times 10^{-4}\ ft \\ 7.295 \times 10^{-5}\ rad \end{Bmatrix}$$

$$\vec{F}^1 = k^1_{4x4}\vec{d}^1_{4x1} - \vec{f}^1_{4x1} = \begin{Bmatrix} -19.05\ kip \\ 0.016\ kip \cdot ft \\ 13.61\ kip \\ -86.467\ kip \cdot ft \end{Bmatrix}$$

b) Element 2

$$\vec{d}^2_{4x1} = \begin{Bmatrix} 5.524 \times 10^{-4}\ ft \\ 7.295 \times 10^{-5}\ rad \\ 5.947 \times 10^{-4}\ ft \\ -6.193 \times 10^{-5}\ rad \end{Bmatrix}$$

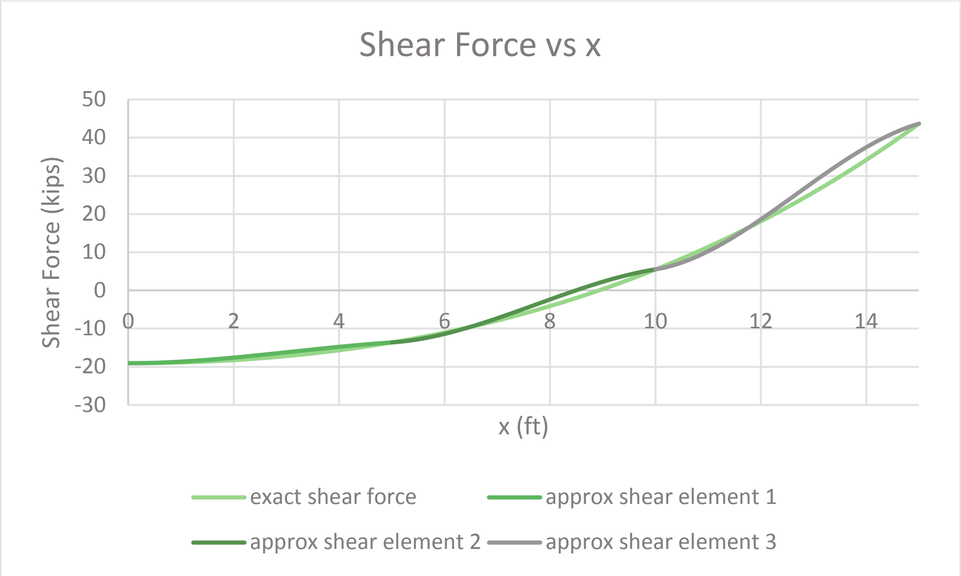
$$\vec{F}^2 = k^2_{4x4}\vec{d}^2_{4x1} - \vec{f}^2_{4x1} = \begin{Bmatrix} -13.61\ kip \\ 86.491\ kip \cdot ft \\ -5.489\ kip \\ -113.475\ kip \cdot ft \end{Bmatrix}$$

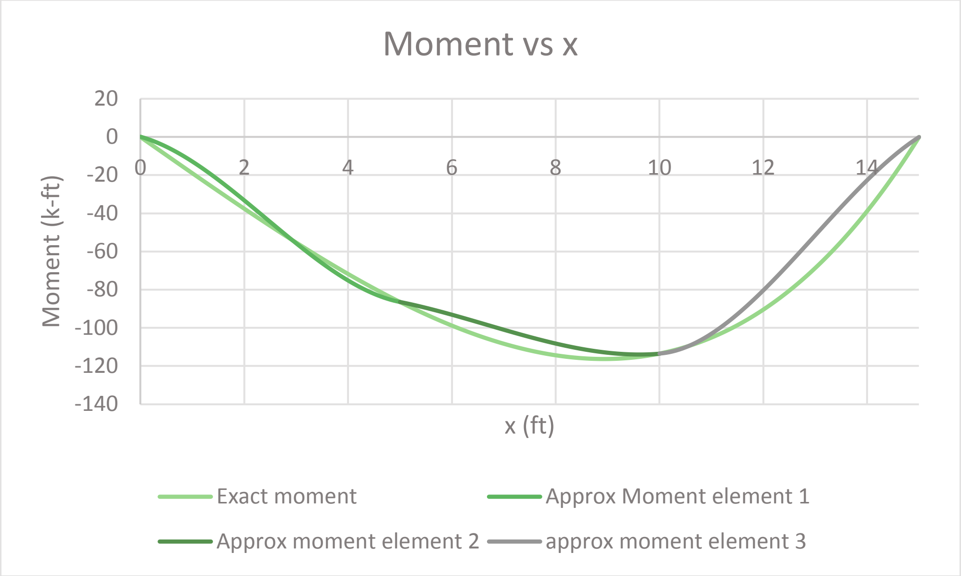
c) Element 3

$$\vec{d}^3_{4x1} = \begin{Bmatrix} 5.947 \times 10^{-4}\ ft \\ -6.193 \times 10^{-5}\ rad \\ 0 \\ -1.527 \times 10^{-4} rad \end{Bmatrix}$$

$$\vec{F}^3 = k^3_{4x4}\vec{d}^3_{4x1} - \vec{f}^3_{4x1} = \begin{Bmatrix} 5.474\ kip \\ 113.5\ kip \cdot ft \\ -43.64\ kip \\ -0.033\ kip \cdot ft \end{Bmatrix}$$

Plotting the approximations to the internal shear and bending moment vs the exact solution we obtain:





III. Discussion

The astute reader will note that we have employed cubic polynomials to approximate the transverse displacements. Such approximations should lead to internal bending moments that are at best linear within each element and internal shear forces that are constant within each element. The process of using the nodal values for the resultant forces to obtain approximations to the bending moment and internal shear force within each element is a form of "stress averaging" that is used as a post processing step in most FEM programs.

Unfortunately such "stress averaging" can make the results of a FEM analysis appear to be more precise than is the case since it hides the sudden jump in values for the internal forces from element to element.

It is also important to note that in general a FEM approximation will typically overestimate the stiffness of the structure. In the numerical example illustrated above, the maximum displacement from the exact solution is underestimated by the FEM approximation. Although not investigated in the current article, it should be expected that as the number of elements are increased better approximations to the displacements, moments, and shears should be expected. The downside to increasing the number of elements is that the number of equations to be solved increases.

Finally, to extend the current discussion to 2D frames and 3D frames one must include the effects of axial forces. For 3D frames, torsion is an additional force resultant that needs to be considered. Future articles in the series will extend the current formulation to include the effects of shear deformations, i.e. Timoshenko beams, and the effects of axial forces before considering the application of column-beam frames.

IV. References

1. Hughes, T.J.R., "The Finite Element Method-Linear Static and Dynamic Finite Element Analysis," Prentice-Hall, Inc., copyright 1987.

2. Bathe, K-J, "Finite Element Procedures," Prentice-Hall, Inc., copyright 1996.